

Twists of quantum groups and noncommutative field theory ¹

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Abstract

The role of quantum universal enveloping algebras of symmetries in constructing a noncommutative geometry of space-time and corresponding field theory is discussed. It is shown that in the framework of the twist theory of quantum groups, the noncommutative (super)space-time defined by coordinates with Heisenberg commutation relations, is (super)Poincaré invariant, as well as the corresponding field theory. Noncommutative parameters of global transformations are introduced.

One of attempts to study the structure of space-time at Planck scale is related with a possible noncommutative nature of space-time, hence, with a noncommutative geometry (for references see e.g. [1, 2]). In this paper we would like to draw attention to interrelations between noncommutative quantum field theories and quantum groups [3]. Recently, an active research takes place in noncommutative field theory related to noncommutative geometry (see the reviews [4, 5] and references therein). One source of examples of noncommutative geometry is the theory of quantum groups [6, 7, 8]. The reason for this is that the latter are, loosely speaking, deformations of Lie groups, which provide numerous geometric structures. There are corresponding structures in quantum groups (QG), where the commutative algebra of functions $F(G)$ on a Lie group G is deformed into an appropriate noncommutative algebra $F_q(G)$, which is defined e.g. by generators and relations [7]. Homogeneous spaces are also subject to deformation, for example $SL(2) \rightarrow SL_q(2)$ or $SU(2) \rightarrow SU_q(2)$ and two-dimensional plane $(x, y) \rightarrow$ “quantum plane” $(x, y)_q$, or two dimensional sphere to the Podles q -sphere $(x, y, z) \rightarrow (x, y, z)_q$. It has been observed by several authors (see e.g. [3, 9, 10, 11]) that the twist theory of quantum groups provides a very useful tool for constructing non-commutative geometry of space-time, including vector bundles, measure, and equations of motion and their solutions.

The most important space of relativistic theory is four-dimensional Minkowski space-time \mathcal{M} , with coordinates x^μ , and with the Poincaré algebra acting on x^μ . To construct NC field theory, the commutative algebra of functions $C(\mathcal{M})$ on \mathcal{M} is deformed to a noncommutative (NC) algebra $C_\theta(\mathcal{M})$. This algebra is generated by NC coordinates x^μ , and probably the simplest relations among the x^μ are

$$[\hat{x}^\mu, \hat{x}^\nu] = \hat{x}^\mu \hat{x}^\nu - \hat{x}^\nu \hat{x}^\mu = i\theta^{\mu\nu}, \quad (1)$$

with a constant antisymmetric matrix θ (see [4, 5]).

There are many possible commutation relations (CR) for x^μ, x^ν with the right hand side linear or quadratic in x^μ (see [12, 13]). However, those written above follow from a special limit of string theory [14] and have attracted substantial interest.

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To construct a field theory on noncommutative space-time with CR (1) for the coordinates, one has to substitute the commutative algebra of fields (functions on \mathcal{M}) by the noncommutative algebra C_θ . In the case of the CR (1) there is a Weyl-Moyal correspondence between these algebras through the Fourier transform. It maps a smooth function $\varphi(x) \in C(\mathcal{M})$ to an element of the algebra C_θ ,

$$\varphi(\hat{x}) = \frac{1}{(2\pi)^4} \int d^4k \tilde{\varphi}(k) \exp(ik\hat{x}), \quad (2)$$

with $\tilde{\varphi}(k)$ being the Fourier transform of the function $\varphi(x)$, and $k\hat{x} = k_\mu \hat{x}^\mu$

$$\tilde{\varphi}(k) = \int d^4x \varphi(x) e^{-ikx}.$$

Then the noncommutative product in the algebra C_θ is

$$\begin{aligned} \varphi(\hat{x})g(\hat{x}) &= \int \frac{dk_1}{(2\pi)^4} \frac{dk_2}{(2\pi)^4} \tilde{\varphi}(k_1) \tilde{g}(k_2) e^{ik_1\hat{x}} e^{ik_2\hat{x}} \\ &= \int \frac{dk_1}{(2\pi)^4} \frac{dk'_2}{(2\pi)^4} \tilde{\varphi}(k_1) \tilde{g}(k'_2 - k_1) e^{-i\theta(k_1, k'_2)} e^{ik'_2\hat{x}}, \end{aligned} \quad (3)$$

where the notation $\theta(k, p) := \frac{1}{2}\theta^{\mu\nu} k_\mu p_\nu$ is introduced for the antisymmetric quadratic form.

Interpreting the convolution of $\tilde{\varphi}(k_1)$ and $\tilde{g}(k_2)$ with the weight function $\exp(-i\theta(k_1, k_2))$ as the Fourier transform of a new product ($*$ -product) of the elements $\varphi(x), g(x) \in C(\mathcal{M})$ one gets

$$\varphi(x) * g(x) = \int \frac{dk_1}{(2\pi)^4} \frac{dk'_2}{(2\pi)^4} \tilde{\varphi}(k_1) \tilde{g}(k'_2 - k_1) \sum_n \frac{1}{n!} (-i\theta(k_1, k'_2))^n e^{ik'_2x}. \quad (4)$$

It is not difficult to check that this $*$ -product on $C(\mathcal{M})$ is still associative, albeit noncommutative. The exponential function $\exp(ik\hat{x})$ generates symmetrized $*$ -products of \hat{x}^ν , which coincide with the usual products of commutative x^ν . Let us point out that the “ $*$ -product” is a general notion of deformation quantization (see the review [15]).

It follows that a field theory on the NC space-time can be constructed using fields $\varphi(x) \in C(\mathcal{M})$, but with multiplication given by the $*$ -product. To fix an action one needs a linear functional on C_θ , and it is represented as an integral on $C(\mathcal{M})$ of the usual form, e.g.

$$S[\varphi] = \int dx \left\{ \frac{1}{2} (\partial_\mu \varphi(x))^2 - \frac{m^2}{2} (\varphi(x))^2 - \frac{\lambda^2}{4!} (\varphi(x))_*^4 \right\}. \quad (5)$$

The integral of the $*$ -product of several functions is invariant only under the cyclic permutations, similarly to the trace of operators:

$$\int dx f_1(x) * f_2(x) * \dots * f_n(x) = \int dx f_2(x) * \dots * f_n(x) * f_1(x).$$

With the $*$ -product chosen (4) one has $\int dx f_1(x) * f_2(x) = \int dx f_1(x) \cdot f_2(x)$, and NC field theory and ordinary field theory coincide on the free field level (the action with quadratic terms only).

However, the interaction term being written as the $*$ -product of the fields, describes a nonlocal interaction, e.g. for the φ_*^3 -theory

$$\begin{aligned} \int dx (\varphi(x))_*^3 &= \int \prod_a \left(\frac{dk_a}{(2\pi)^4} \tilde{\varphi}(k_a) \right) \exp(-i \sum_{b<c} \theta(k_b, k_c)) \delta(\sum_j k_j) \\ &= \int \prod_a dx_a \varphi(x_1) \varphi(x_2) \varphi(x_3) \exp(2i(x_1 - x_3) \theta^{-1}(x_2 - x_3)), \end{aligned}$$

provided that the matrix θ is invertible (or one has to restrict the arguments to those x^j for which θ^{ij} has an inverse).

Quantization of the scalar field theory with the action $S[\varphi]$ by path integral methods yields the standard perturbation theory, but the interaction vertices include an extra oscillating factor,

$$V(k_1, \dots, k_4) = \frac{\lambda^2}{4!} \delta(\sum_a k_a) \prod_{b<c} e^{-\frac{i}{2} \theta_{\mu\nu} k_b^\mu k_c^\nu}.$$

This factor has only cyclic symmetry (due to the delta-function) and results in different contributions as compared to local QFT, and even in a different structure of the Feynman diagrams (planar versus non-planar graphs). The diagrammatic analysis of unitarity yields a condition on $\theta^{\mu\nu}$: $\theta^{0j} = 0$. Thus the time coordinate commutes with the space coordinates, and one can apply the Hamiltonian formalism for the action (5).

Reformulating NC space-time field theory as a usual one (5) with a nonlocal interaction, it is possible to apply standard techniques to quantize it. An obvious drawback is the appearance of the set of constants $\theta^{\mu\nu}$ breaking the Lorentz invariance: $x^\mu \rightarrow \Lambda^\mu_\nu x^\nu$, $\theta^{\mu\nu} \rightarrow \Lambda^\mu_\alpha \Lambda^\nu_\beta \theta^{\alpha\beta} = \tilde{\theta}^{\mu\nu} \neq \theta^{\mu\nu}$. To cure this problem we propose to use a quantum group technique.

In this discussion we need such objects from the theory of quantum groups as a Hopf algebra \mathcal{H} , its \mathcal{H} -module algebra \mathcal{A} , \mathcal{H} -modules and \mathcal{A} -modules V, W (linear spaces for \mathcal{H} - and \mathcal{A} -representations). At the same time these objects have a physical interpretation: \mathcal{H} is the symmetry algebra of the system under consideration, \mathcal{A} is the algebra of observables, and their representation space is the space of states of the system. There are also additional structures, such as a $*$ -operation (real form), a scalar product etc., which will be introduced later.

The symmetry of the relativistic field theory is described by the universal enveloping algebra $\mathcal{U}(\mathcal{P})$ of the Poincaré Lie algebra \mathcal{P} with generators of translations P_μ and rotations $M_{\mu\nu}$:

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha} M_{\nu\beta} - \eta_{\mu\beta} M_{\nu\alpha} - \eta_{\nu\alpha} M_{\mu\beta} + \eta_{\nu\beta} M_{\mu\alpha}), \\ [M_{\mu\nu}, P_\alpha] &= -i(\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu). \end{aligned} \tag{6}$$

The essential part of the Hopf algebra structures $\mathcal{H}(m, \Delta, \gamma, \epsilon)$ (see [8, 6] for details) is given by the associative product (with the commutation relations (6) for $\mathcal{U}(\mathcal{P})$ in our case) and by a coproduct map $\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ defining an action of the Hopf algebra \mathcal{H} in the tensor product of two (or more) of its representations. The action of the generators $Y \in \mathcal{P}$ in a tensor product $V \otimes W$ is given by the symmetric map (coproduct) $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$, or

$$\Delta(Y)(v \otimes w) = (\hat{Y}v) \otimes w + v \otimes (\hat{Y}w), \tag{7}$$

where the hat means the action of a Hopf algebra element in the corresponding representation space. There are two other maps in the definition of the Hopf algebra: the counit $\epsilon : \mathcal{H} \rightarrow \mathcal{C}$ (a one-dimensional representation of \mathcal{H}) and the antipode $\gamma : \mathcal{H} \rightarrow \mathcal{H}$, which is an algebra antihomomorphism. These maps are subject to quite a few axioms, of course [6, 7, 8]. On the generators of $\mathcal{U}(\mathcal{P})$ the antipode and counit are: $\gamma(Y) = -Y$, $\epsilon(Y) = 0$, $\epsilon(1) = 1$.

There is a useful transformation (twist) of the structure maps of a Hopf algebra, which is an equivalence relation among Hopf algebras, preserving their category of representations. This transformation $\mathcal{H} \rightarrow \mathcal{H}_t$ is realized by an invertible twist element [16]

$$\mathcal{F} = \sum_i f_1^i \otimes f_2^i \in \mathcal{H} \otimes \mathcal{H}.$$

It does not change the multiplication in \mathcal{H} , but transforms the coproduct according to

$$\Delta(h) \rightarrow \Delta_t(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \quad h \in \mathcal{H}.$$

This similarity transformation preserves the coassociativity of the twisted coproduct if \mathcal{F} satisfies the following twist equation (two-cocycle condition) in $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ [16]

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F}, \quad (\epsilon \otimes id)\mathcal{F} = 1 \otimes 1, \quad (8)$$

where \mathcal{F}_{23} means $\sum_i 1 \otimes f_1^i \otimes f_2^i \in \mathcal{H}^{\otimes 3}$, and $(\Delta \otimes id)\mathcal{F} := \sum_i \Delta(f_1^i) \otimes f_2^i \in \mathcal{H}^{\otimes 3}$. The twist does not change the counit homomorphism, but similarity-transforms the antipode:

$$\gamma(Y) \rightarrow \gamma_t(Y) = u\gamma(Y)u^{-1}, \quad \text{where} \quad u = \sum_i f_1^i \cdot \gamma(f_2^i) \in \mathcal{H}. \quad (9)$$

Usually the twist element is not symmetric under the permutation of its tensor factors: $\mathcal{F} \neq \mathcal{F}_{21} = \sum_i f_2^i \otimes f_1^i$. Hence, the twisted coproduct $\Delta_t(h) := \sum h_{(1)} \otimes h_{(2)}$ is also non-symmetric

$$\Delta_t(h) \neq \Delta_t^{op}(h) = \sum h_{(2)} \otimes h_{(1)}.$$

However, for the quantum group case the coproduct $\Delta_t(h)$ and the opposite coproduct $\Delta_t^{op}(h)$ are related by a similarity transformation with the \mathcal{R} -matrix:

$$\mathcal{R}\Delta_t = \Delta_t^{op}\mathcal{R}, \quad \mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2 \in \mathcal{H} \otimes \mathcal{H}.$$

In our case, starting with the symmetric coproduct (7) the \mathcal{R} -matrix is given by $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$.

There are well-known statements from the theory of quantum groups which will be used in our discussion of a particular case of noncommutative space-time. Having an action of \mathcal{H} on an associative algebra \mathcal{A} with consistency of the coproduct of \mathcal{H} and multiplication of \mathcal{A} (a Leibniz rule),

$$\hat{h}(a \cdot b) = \sum (\hat{h}_{(1)}a) \cdot (\hat{h}_{(2)}b),$$

the multiplication in \mathcal{A} has to be changed after twisting $\mathcal{H} \rightarrow \mathcal{H}_t$ to preserve this consistency. The new product in \mathcal{A}_t is

$$a * b = \sum (\hat{f}_1^i a) \cdot (\hat{f}_2^i b), \quad a, b \in \mathcal{A}_t, \quad (10)$$

where a notation was introduced for $\mathcal{F}^{-1} := \sum \bar{f}_1^i \otimes \bar{f}_2^i$, and the action (representation) of elements from \mathcal{H} on elements from \mathcal{A}_t is the same as before twisting.

The product $\varphi(x_1^\mu) * \varphi(x_2^\nu)$ of quantum fields with independent arguments belongs to the tensor product of two copies of the algebra $C_\theta(\mathcal{M})$. After twisting of \mathcal{H} the elements of different copies of $\mathcal{A} \otimes \mathcal{A}$ will not commute:

$$(a_1 \otimes 1)(1 \otimes a_2) = (a_1 \otimes a_2), \quad \text{but}$$

$$(1 \otimes a_2)(a_1 \otimes 1) = (\hat{\mathcal{R}}_2 a_1) \otimes (\hat{\mathcal{R}}_1 a_2) \neq (a_1 \otimes a_2), \quad \forall a_1, a_2 \in \mathcal{A}. \quad (11)$$

(Recall that the hat indicates the action of Hopf algebra elements on the relevant representation spaces.) If in addition one has an \mathcal{H} -covariant representation of the algebra \mathcal{A} in a vector space V , then the action of the elements of \mathcal{A} on vectors of V will be changed correspondingly $a \cdot v \rightarrow a * v = \sum (\hat{f}_1^i a) \cdot (\hat{f}_2^i v)$.

It is important that real forms survive a twist. Recall that a $*$ -operation (real form) on a Hopf algebra \mathcal{H} means an antilinear involutive algebra anti-automorphism and coalgebra automorphism. Due to the uniqueness of the antipode, the identity $\gamma * = * \gamma^{-1}$ is always valid, and one can re-define the real form as $\gamma^{2n} *$ for any integer number n . We can also consider homomorphic and anti-cohomomorphic antilinear operations of the kind $\xi = \gamma^{2n+1} *$.

To ensure consistency between real forms and the action of \mathcal{H} on some \mathcal{H} -module algebra \mathcal{A} with anti-involution $a \rightarrow \bar{a}$, one has to require $\overline{(ha)} = \gamma(h^*)\bar{a}$, for $h \in \mathcal{H}$ and $a \in \mathcal{A}$. So by the real form of a quantum algebra we will mean a homomorphic and anti-cohomomorphic antilinear involution $\xi = \gamma \circ *$.

Twisting a Hopf algebra $\mathcal{H} \rightarrow \mathcal{H}_t$ the same $*$ -operation is defined on \mathcal{H}_t if the twist \mathcal{F} satisfies the condition

$$\mathcal{F}^* = \sum f_1^* \otimes f_2^* = \mathcal{F}^{-1} = \sum \bar{f}_1 \otimes \bar{f}_2. \quad (12)$$

For the involution ξ the analogous natural requirement is [9]

$$(\xi \otimes \xi)\mathcal{F} = \tau(\mathcal{F}) := \mathcal{F}_{21} = \sum f_2 \otimes f_1, \quad (13)$$

where τ is the permutation of the factors in $\mathcal{H} \otimes \mathcal{H}$.

Suppose now that \mathcal{A} possesses a measure μ , i.e. a linear functional positive on elements of the form $a \cdot \bar{a}$ (like the function algebra on a locally compact topological space does). The same measure is valid for \mathcal{A}_t , for these \mathcal{H} -module algebras \mathcal{A} and \mathcal{A}_t coincide as linear spaces [9]. Indeed, we find $a * \bar{a} = \bar{f}_1 a \cdot \bar{f}_2 \bar{a} = \bar{f}_1 a \cdot \overline{(\xi(\bar{f}_2)a)}$. If identity (13) is fulfilled, the relation $\bar{f}_1 \otimes \xi(\bar{f}_2) = \xi(\bar{f}_2) \otimes \bar{f}_1$ holds as well and, consequently, $\bar{f}_1 \otimes \xi(\bar{f}_2)$ can be represented by a sum $\sum \varphi_i \otimes \varphi_i$. Further, we have $a * \bar{a} = \sum \varphi_i a \cdot \overline{\varphi_i a}$, and therefore $\mu(a * \bar{a}) \geq 0$. In case that (12) is true, one can extend the Hopf algebra by adding the square root of the element u that was introduced in (9). It is straightforward that the composition of the coboundary twist with the element $\Delta(u^{-\frac{1}{2}})(u^{\frac{1}{2}} \otimes u^{\frac{1}{2}})$ and successive twist with the element $(u^{-\frac{1}{2}} \otimes u^{-\frac{1}{2}})\mathcal{F}^{-1}(u^{\frac{1}{2}} \otimes u^{\frac{1}{2}})$ obeys (13). This double transformation is carried out by means of the 2-cocycle $\Delta(u^{-\frac{1}{2}})\mathcal{F}^{-1}(u^{\frac{1}{2}} \otimes u^{\frac{1}{2}})$, and the required property (13) readily follows from (12) and the identity $(u \otimes u)\tau(\gamma \otimes \gamma)(\mathcal{F}^{-1}) = \mathcal{F}\Delta(u)$ fulfilled for any solution to the twist equation [16] (the element u is exactly the same

as the one taking part in the definition of the twisted antipode (9)). So we can apply all the previous considerations to this composite twist, which differs from initial one by an inner automorphism only.

Let's deform the Poincaré algebra $\mathcal{U}(\mathcal{P})$ as a Hopf algebra by a simple twist element depending only on the generators of translations P_μ (an abelian subalgebra of \mathcal{P}) [3]:

$$\mathcal{F} = \exp \left(\frac{i}{2} \theta^{\mu\nu} P_\mu \otimes P_\nu \right) \quad (14)$$

with a constant matrix $\theta^{\mu\nu}$ (we take it to be real and antisymmetric). As an associative algebra $\mathcal{U}_t(\mathcal{P})$ is not changed (we have the same commutation relations of generators $M_{\mu\nu}, P_\alpha$) nor is the coproduct of P_α : $\Delta_t(P_\alpha) = \Delta(P_\alpha)$. However, the coproduct of $M_{\mu\nu}$ is changed:

$$\begin{aligned} \Delta_t(M_{\mu\nu}) &= \text{Ad} \left(\exp \left(\frac{i}{2} \theta^{\alpha\beta} P_\alpha \otimes P_\beta \right) \right) \Delta(M_{\mu\nu}) \\ &= \Delta(M_{\mu\nu}) - \frac{1}{2} \theta^{\alpha\beta} ((\eta_{\alpha\mu} P_\nu - \eta_{\alpha\nu} P_\mu) \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu} P_\nu - \eta_{\beta\nu} P_\mu)). \end{aligned} \quad (15)$$

It was already mentioned that the coproduct defines an action of the Hopf algebra on the product of elements from \mathcal{A} , and the product of \mathcal{A} is also changed accordingly, to be consistent with Δ_t . The algebra $\mathcal{C}(\mathcal{M})$ is generated by the x^μ , and after twisting $\mathcal{C}(\mathcal{M}) \rightarrow \mathcal{C}_t(\mathcal{M})$ the new product is

$$\begin{aligned} x^\mu * x^\nu &= \sum (\hat{f}_1 x^\mu)(\hat{f}_2 x^\nu) \\ &= \sum_{k=0}^{\infty} \frac{(i/2)^k}{k!} \prod_{j=1}^k \theta^{\mu_j, \nu_j} (\partial_{\mu_1} \dots \partial_{\mu_k} x^\mu) (\partial_{\nu_1} \dots \partial_{\nu_k} x^\nu) \\ &= x^\mu x^\nu + \frac{i}{2} \theta^{\mu\nu}. \end{aligned} \quad (16)$$

Hence,

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}, \quad (17)$$

and this yields $\mathcal{C}_t(\mathcal{M}) = \mathcal{C}_\theta$. One can check that with the deformed coproduct (15) these CR are invariant under the action of $M_{\mu\nu}$ [3].

One has to mention that the interpretation of the Weyl - Moyal product using abelian twist has been known for some time in the deformation quantization (see [17, 18] and references therein).

It is possible to write explicitly the star product of x^μ and $\phi(x)$ according to (16)

$$x^\mu * \phi(x) = x^\mu \phi(x) + \frac{i}{2} \theta^{\mu\nu} \partial_\nu \phi(x). \quad (18)$$

Hence, the $*$ -multiplication by x^μ can be represented as an action by the operator $x^\mu + \frac{i}{2} \theta^{\mu\nu} \partial_\nu$ in the space $C(\mathcal{M})$. Similarly, the star product of $\varphi(x)$ and $g(x)$ can be represented as an action of a differential operator on the second factor

$$\varphi(x) * g(x) = \left(\varphi(x) + \sum_{k=1}^{\infty} \frac{(i/2)^k}{k!} \prod_{j=1}^k \theta^{\mu_j, \nu_j} (\partial_{\nu_1} \dots \partial_{\nu_k} \varphi(x)) (\partial_{\mu_1} \dots \partial_{\mu_k}) \right) g(x). \quad (19)$$

However, one can use instead of usual functions on Minkowski space-time with the star product the algebra generated by noncommutative \hat{x}^μ . Correspondence between the bases of these algebras in our case (1), (17) is given by the generating functions of their bases $\exp(ikx)$ and $\exp(ik\hat{x})$ (cf (3)). Then one can express the action of the symmetry algebra generators in terms of \hat{x}^μ and derivatives (see [1, 19]).

The action of momentum generators P_μ on classical and quantum fields $\varphi(x)$ is supposed to be the same

$$P_\mu \varphi(x) = i \frac{\partial}{\partial x^\mu} \varphi(x).$$

However, in classical theory fields are given by different smooth functions as elements of $\mathcal{C}(\mathcal{M})$ with Fourier expansion (2) and the generators P_μ are realized as partial derivatives $P_\mu = i\partial/\partial x^\mu$. In quantum theory $\varphi(x)$ and P_μ are fixed operators as elements of the algebra of observables \mathcal{A} . The action of P_μ on $\varphi(x)$ is defined by the commutator

$$P_\mu \cdot \varphi(x) = [P_\mu, \varphi(x)],$$

and having in mind the expansion of $\varphi(x)$ in terms of the creation and annihilation operators $a(k)$, $a^\dagger(p)$, one gets

$$[P_\mu, a(k)] = -k_\mu a(k). \quad (20)$$

We could apply the twisting of the Poincaré algebra with the action (20) after quantizing field theory [20].

Using the twist element $\mathcal{F} = \exp(\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu)$, we have to change product of observables according to the general rule

$$\begin{aligned} a * b &= m \circ (e^{-\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu})(a \otimes b) \\ &= m \circ \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{2} \right)^n \prod_{j=1}^n \theta^{\mu_j \nu_j} adP_{\mu_j} \otimes adP_{\nu_j} \right) (a \otimes b). \end{aligned} \quad (21)$$

Hence, the twisted products of the creation and annihilation operators are related with the standard products as follows

$$a(k) * a(p) = a(k)a(p)e^{-i\theta(k,p)} \quad (22)$$

$$a(k) * a^\dagger(p) = a(k)a^\dagger(p)e^{i\theta(k,p)}. \quad (23)$$

One can see that the $*$ -products of the exponentials (4) and annihilation or creation operators (22) have the same factors. Being expressed in terms of the twisted product, the commutation relations are

$$\begin{aligned} a(k) * a(p) &= a(p) * a(k)e^{-2i\theta(k,p)} \\ a(k) * a^\dagger(p) - e^{2i\theta(k,p)} a^\dagger(p) * a(k) &= \delta(k-p), \end{aligned} \quad (24)$$

where $\theta(k,p) = -\theta(p,k) = \frac{1}{2}\theta^{\mu\nu} k_\mu p_\nu$. The relations (24) reproduce a scalar Zamolodchikov-Faddeev algebra. Noncommutativity of both coordinates x^μ and the operators $a(k)$, $a^\dagger(k)$ was

introduced in [21] with a phase in (24) inconsistent with the twisted Poincaré algebra (see detailed explanation in [22]).

The parameters $\Lambda_\nu^\mu(\omega)$, a^μ of the global Poincaré transformations generate the algebra of functions $F(G)$ on the Poincaré group G . This commutative algebra $F(G) \simeq (\mathcal{U}(\mathcal{P}))^*$ is dual to $\mathcal{U}(\mathcal{P})$, and after twisting $\mathcal{U}(\mathcal{P})$ the product of the dual Hopf algebra $(\mathcal{U}(\mathcal{P}))^*$ is changed.

An important object connecting a pair of dual Hopf algebras is the canonical element (a bicharacter) [6]

$$\mathcal{T} = \sum e_k \otimes e^k, \quad e_k \in \mathcal{H}^*, \quad e^k \in \mathcal{H}, \quad \langle e_k, e^m \rangle = \delta_k^m,$$

where e_k and e^m are dual linear bases of \mathcal{H}^* and \mathcal{H} . Here we have (using a short hand notation $a^\mu \otimes P_\mu := a^\mu P_\mu$ etc)

$$\mathcal{T} = \exp(ia^\mu P_\mu) \exp(i\omega^{\mu\nu} M_{\mu\nu}).$$

In the case of the twist (14) the generators $\omega^{\mu\nu}$ or $\Lambda_\nu^\mu(\omega)$ are the same (commutative), but the a^μ become noncommutative (see [10, 23, 24]),

$$[a^\mu, a^\nu] = i\theta^{\mu\nu} - i\Lambda_\alpha^\mu \Lambda_\beta^\nu \theta^{\alpha\beta}. \quad (25)$$

This can be obtained from the RTT-relations [7] using the matrix representation of $\mathcal{U}(\mathcal{P})$ and the R -matrix, or from the general recipe (10) using the $\mathcal{U}(\mathcal{P})$ -bimodule structure of $(\mathcal{U}(\mathcal{P}))^*$. Due to the commutativity of $\Lambda(\omega)$, in the representations V of $(\mathcal{U}_t(\mathcal{P}))^*$ with $\Lambda_\alpha^\mu = \delta_\alpha^\mu$, the generators a^μ are commutative in such V . However, if $\Lambda_\alpha^\mu \neq \delta_\alpha^\mu$, then a^μ are noncommutative and can not be put to zero.

The transformation of the coordinates x^μ is given by the coaction $\delta : C_\theta \rightarrow F_\theta(G) \otimes C_\theta$,

$$\tilde{x}^\mu := \delta(x^\mu) = \Lambda_\alpha^\mu \otimes x^\alpha + a^\mu \otimes 1. \quad (26)$$

The transformed generators satisfy the same relations, $[\tilde{x}^\mu, \tilde{x}^\nu] = i\theta^{\mu\nu}$. Hence one can conclude that the noncommutative space-time (1) is invariant under the twisted Poincaré algebra $\mathcal{U}_t(\mathcal{P})$. The action of the algebra generators can be integrated

$$\frac{df}{d\tau} = \alpha_j Y_j f \rightarrow f(\tau) = \exp(\alpha_j Y_j \tau) f(0).$$

However, to have a similar action on the star product of two elements f, g the parameters α_j have to be noncommutative (see (25)).

Tensoring two copies of the NC space-time algebra, $\mathcal{C}_\theta \otimes \mathcal{C}_\theta$ with generators $x_1^\mu = x^\mu \otimes 1$ and $x_2^\nu = 1 \otimes x^\nu$, one gets their commutation relations according to (11) with R-matrix $\mathcal{R} = \exp(-i\theta^{\mu\nu} P_\mu \otimes P_\nu)$ [24]:

$$\begin{aligned} x_1^\mu x_2^\nu - x_2^\nu x_1^\mu &:= x^\mu \otimes x^\nu - (1 \otimes x^\nu)(x^\mu \otimes 1) \\ &= x^\mu \otimes x^\nu - \sum_{k=0}^{\infty} \frac{(i)^k}{k!} \prod_{j=1}^k \theta^{\mu_j, \nu_j} (\partial_{\nu_1} \dots \partial_{\nu_k} x^\mu) \otimes (\partial_{\mu_1} \dots \partial_{\mu_k} x^\nu) \\ &= x^\mu \otimes x^\nu - x^\mu \otimes x^\nu - i\theta^{\mu\nu} = i\theta^{\mu\nu}. \end{aligned} \quad (27)$$

This property results in an extra factor in the Fourier transform of the vacuum expectation value $\langle \varphi(x_1) * \varphi(x_2) * \dots * \varphi(x_n) \rangle$ of quantum fields [10]. Application of the abelian twist to gauge field theories are also considered (see [25] and references therein).

Similar arguments as in [3] can be applied in the case of (extended) supersymmetry and of the Poincaré superalgebra $s\mathcal{P}$ with additional supercharges (odd generators) $Q_\alpha, \bar{Q}_{\dot{\beta}}$ to get a noncommutative superspace as in [26]. The Poincaré Lie superalgebra commutation relations (the commutators below are Z_2 -graded, i.e. if both elements are odd it is the anticommutator) are

$$\begin{aligned} [P_\mu, Q_\alpha] &= 0, & [M_{\mu\nu}, Q_\alpha] &= i(\sigma_{\mu\nu})_\alpha^\beta Q_\beta, \\ [Q_\alpha, Q_\beta] &= 0, & [M_{\mu\nu}, \bar{Q}_{\dot{\beta}}] &= i(\bar{\sigma}_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}, \\ [\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}] &= 0, & [Q_\alpha, \bar{Q}_{\dot{\beta}}] &= 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu. \end{aligned}$$

The generators P_μ, Q_α define an abelian (supercommutative) subalgebra, and abelian twists depending on odd generators can be constructed as in the non-graded case, e.g.

$$\mathcal{F} = \exp(C^{\alpha\beta} Q_\alpha \otimes Q_\beta) = 1 + C^{\alpha\beta} Q_\alpha \otimes Q_\beta - \det C Q_1 Q_2 \otimes Q_1 Q_2, \quad (28)$$

with symmetric matrix $C^{\alpha\beta} = C^{\beta\alpha}$. The exponent reproduces a Poisson tensor defining superbrackets (see e.g. [27]), and can be used to construct noncommutative superspace preserving super-Poincaré covariance [28, 29].

The algebraic sector of the twisted Hopf superalgebra $\mathcal{U}_t(s\mathcal{P})$ is not changed, as well as the coproduct of the abelian subalgebra of (super)translations with the generators P_μ, Q_α . However, the coproducts of $M_{\mu\nu}$ and $\bar{Q}_{\dot{\beta}}$ are changed:

$$\begin{aligned} \Delta_t(M_{\mu\nu}) &= \mathcal{F} \Delta(M_{\mu\nu}) \mathcal{F}^{-1} \\ &= \Delta(M_{\mu\nu}) - i\{C^{\alpha\beta}(\sigma_{\mu\nu})_\alpha^\gamma + C^{\alpha\gamma}(\sigma_{\mu\nu})_\alpha^\beta\} Q_\gamma \otimes Q_\beta, \end{aligned}$$

$$\Delta_t(\bar{Q}_{\dot{\gamma}}) = \Delta(\bar{Q}_{\dot{\gamma}}) + 2C^{\alpha\beta} \sigma_{\alpha\dot{\gamma}}^\mu (Q_\beta \otimes P_\mu - P_\mu \otimes Q_\beta).$$

The standard realization of the supercharges

$$Q_\alpha = \partial/\partial\theta^\alpha - i\sigma_{\alpha\dot{\beta}}^\mu \bar{\theta}^{\dot{\beta}} \partial/\partial x^\mu, \quad \bar{Q}_{\dot{\beta}} = \partial/\partial\bar{\theta}^{\dot{\beta}} - i\theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu \partial/\partial x^\mu$$

yields noncommutative generators $x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\beta}}$ of Minkowski superspace $s\mathcal{M}_t$,

$$[\theta^\alpha, \theta^\beta] = 2C^{\alpha\beta}, \quad [x^\mu, \theta^\alpha] = 2iC^{\alpha\beta} \sigma_{\beta\dot{\gamma}}^\mu \bar{\theta}^{\dot{\gamma}}, \quad [x^\mu, x^\nu] = 2C^{\alpha\beta} \sigma_{\alpha\dot{\gamma}}^\mu \sigma_{\beta\dot{\delta}}^\nu \bar{\theta}^{\dot{\gamma}} \bar{\theta}^{\dot{\delta}}.$$

These commutators are consequences of the star products $a*b = m \circ (\exp(-C^{\alpha\beta} Q_\alpha \otimes Q_\beta))(a \otimes b)$

$$\theta^\alpha * \theta^\beta = \theta^\alpha \cdot \theta^\beta + C^{\alpha\beta}, \quad x^\mu * \theta^\alpha = x^\mu \cdot \theta^\alpha + iC^{\alpha\beta} \sigma_{\beta\dot{\gamma}}^\mu \bar{\theta}^{\dot{\gamma}},$$

$$x^\mu * x^\nu = x^\mu \cdot x^\nu + C^{\alpha\beta} \sigma_{\alpha\dot{\gamma}}^\mu \bar{\theta}^{\dot{\gamma}} \sigma_{\beta\dot{\delta}}^\nu \bar{\theta}^{\dot{\delta}}.$$

As in the previous case one can consider realization of the $*$ -multiplication of the standard/usual elements of the algebra of functions $\varphi(x, \theta, \bar{\theta})$ by $\theta^\alpha *$ or by $x^\mu *$ as an action by the operators

$$\theta^\alpha * = \theta^\alpha \cdot + C^{\alpha\beta} (\partial/\partial\theta^\beta - i\sigma_{\beta\dot{\gamma}}^\nu \bar{\theta}^{\dot{\gamma}} \partial/\partial x^\nu), \quad x^\mu * = x^\mu \cdot + iC^{\alpha\beta} \sigma_{\alpha\dot{\gamma}}^\mu \bar{\theta}^{\dot{\gamma}} (\partial/\partial\theta^\beta - i\sigma_{\beta\dot{\delta}}^\nu \bar{\theta}^{\dot{\delta}} \partial/\partial x^\nu).$$

It is important to point out that generators (parameters) of the deformed Poincaré supergroup $(\mathcal{U}_t(s\mathcal{P}))^* : \omega^{\mu\nu}, b^\mu, \lambda^\alpha, \bar{\lambda}^{\dot{\beta}}$ dual to $M_{\mu\nu}, P_\mu, Q_\alpha, \bar{Q}_{\dot{\beta}}$ will be not supercommutative. However, their commutation relations will be different from those of $s\mathcal{M}_t$.

Representing the canonical element \mathcal{T} of the twisted Poincaré superalgebra $\mathcal{U}_t(s\mathcal{P})$ and its dual quantum Poincaré supergroup $(\mathcal{U}_t(s\mathcal{P}))^*$ in the form

$$\mathcal{T} = \exp(i\bar{\lambda}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}) \exp(i\lambda^\alpha Q_\alpha) \exp(ib^\mu P_\mu) \exp(i\omega^{\mu\nu} M_{\mu\nu}), \quad (29)$$

one can get e.g. from the RTT-relation [7] that $\omega^{\mu\nu}, \bar{\lambda}^{\dot{\alpha}}$ are supercommutative while λ^α and b^μ do not super-commute e.g.

$$[\lambda^\alpha, \lambda^\beta] = 2C^{\alpha\beta} - 2(S(\omega))^\alpha_\gamma (S(\omega))^\beta_\delta C^{\gamma\delta}.$$

Let us point out that with the form (29) of the canonical element the generators b^μ are not hermitian. The twist (28) also does not preserve the standard $*$ -operation of the Poincaré superalgebra. One can also use the twist (28) and coproducts of the generators $\omega^{\mu\nu}, b^\mu, \lambda^\alpha, \bar{\lambda}^{\dot{\beta}}$

$$\Delta(\omega^{\mu\nu}) = \mathcal{D}^{\mu\nu}(\omega \otimes 1, 1 \otimes \omega),$$

$$\Delta(\lambda^\alpha) = \lambda^\alpha \otimes 1 + (S(\omega))^\alpha_\beta \otimes \lambda^\beta, \Delta(\bar{\lambda}^{\dot{\alpha}}) = \bar{\lambda}^{\dot{\alpha}} \otimes 1 + (\bar{S}(\omega))^{\dot{\alpha}}_{\dot{\delta}} \otimes \bar{\lambda}^{\dot{\delta}},$$

$$\Delta(b^\mu) = b^\mu \otimes 1 + \Lambda(\omega)^\mu_\nu \otimes b^\nu - 2i(\lambda^\alpha (\bar{S}(\omega))^{\dot{\alpha}}_{\dot{\delta}} \otimes \bar{\lambda}^{\dot{\delta}}) \sigma^\mu_{\alpha\dot{\alpha}}$$

to get corresponding star products. $\mathcal{D}^{\mu\nu}(\omega \otimes 1, 1 \otimes \omega)$ is the BCH series of the Lorentz algebra and $\Lambda(\omega), \bar{S}(\omega), (S(\omega))^\beta_\delta$ are the Lorentz transformation matrices acting on the vector and Weyl spinor indices. The commutation relations of the NC superspace $s\mathcal{M}_t$ are invariant with respect to the twisted Poincaré superalgebra $\mathcal{U}_t(s\mathcal{P})$ and supergroup.

Some consequences of the twisted Poincaré supersymmetry applied to the Wess-Zumino model are discussed in [30].

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